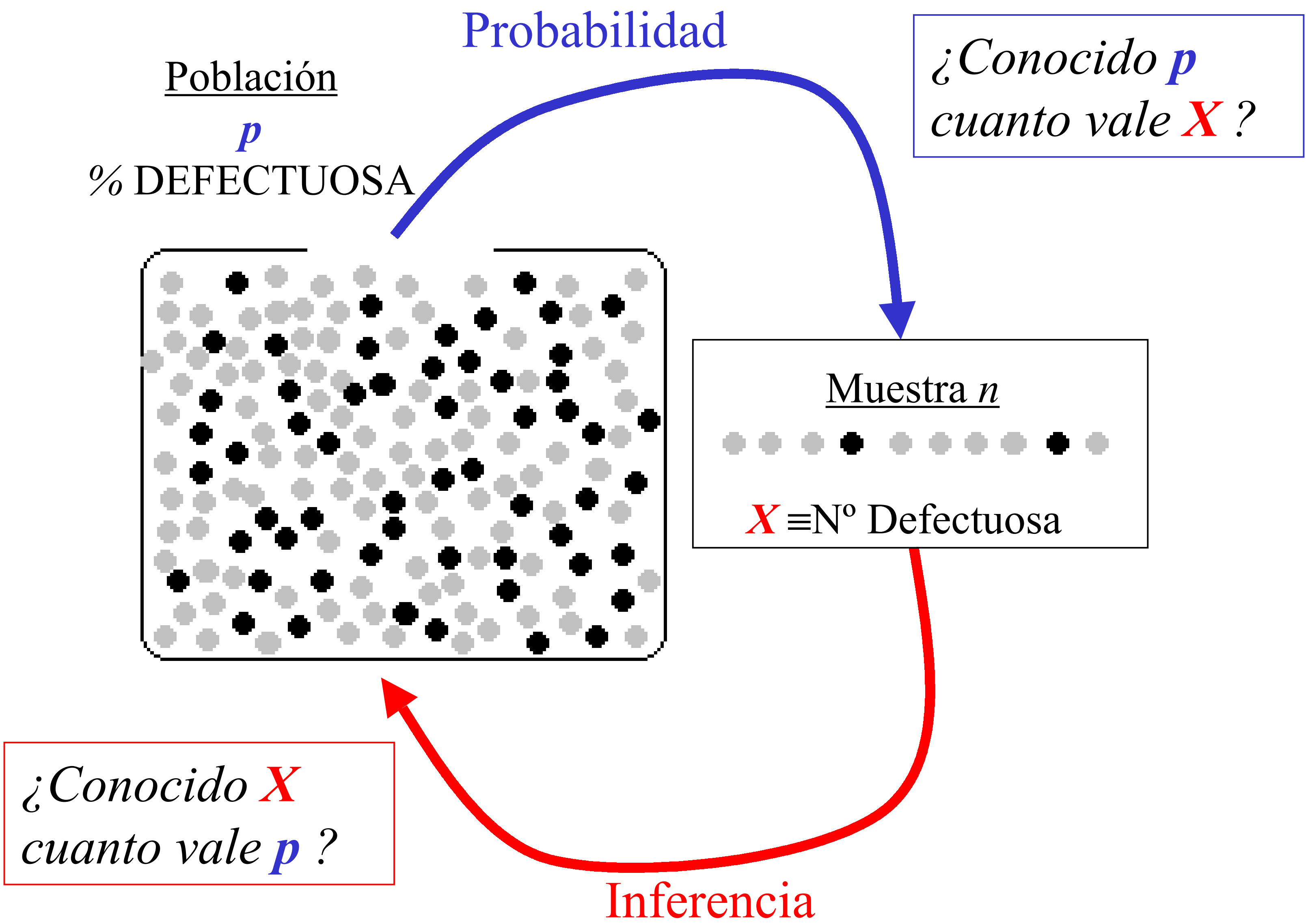
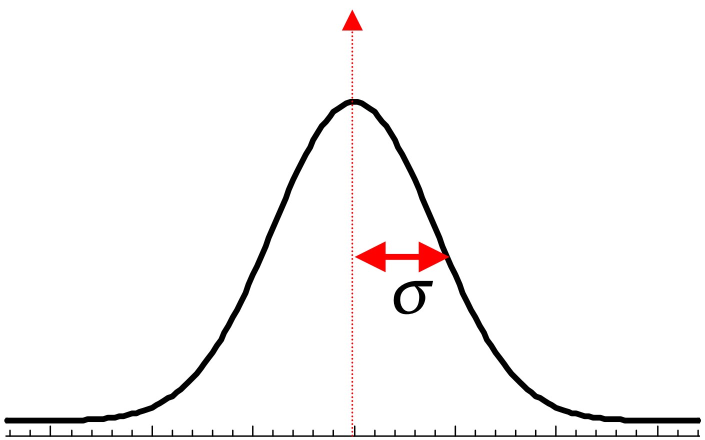
Estimación puntual


Probabilidad
Población
p
% DEFECTUOSA
¿Conocido p
cuanto vale X ?
Muestra n
X ≡Nº Defectuosa
¿Conocido X
cuanto vale p ?
Inferencia

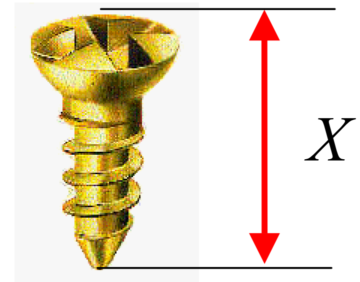
POBLACIÓN



μ

$$X \rightarrow N(\mu, \sigma)$$

$$\boxed{\begin{array}{c} Parámetros \\ ¿\mu, \sigma? \end{array}}$$



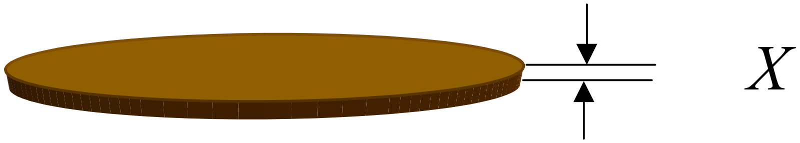
MUESTRA *n*

$$X_1, X_2, \ldots, X_n$$

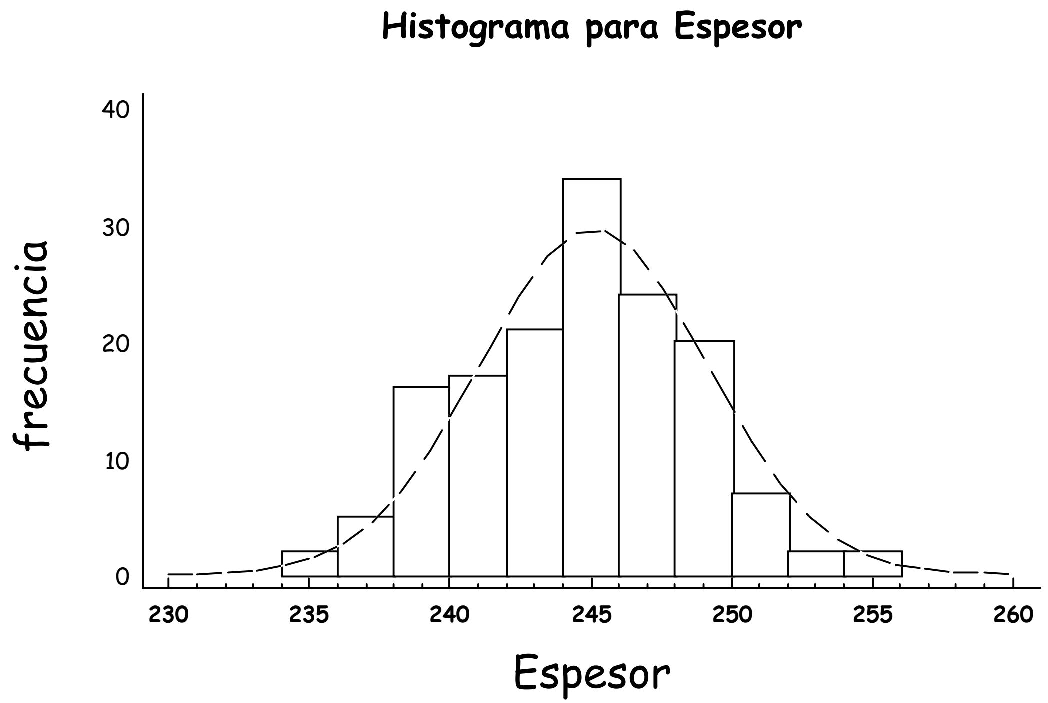
Datos Conocidos

?

Espesores de 150 obleas de Silicio (micras)



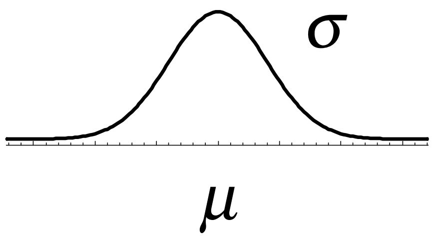
240	235	240	240	247	237	243	242	236	239
243	237	243	242	245	239	245	245	239	240
250	246	244	246	255	242	248	248	241	242
253	249	249	249	250	247	251	251	246	243
248	246	246	248	249	245	250	249	242	244
238	240	245	240	237	242	244	242	243	239
242	241	250	243	239	244	246	245	246	240
245	246	250	246	243	246	246	248	247	250
251	247	247	250	247	251	250	243	252	252
247	249	248	248	246	248	246	246	247	250
239	240	238	241	242	243	241	241	241	241
242	243	240	245	244	245	239	244	243	243
246	244	245	243	245	247	244	245	245	249
250	248	248	247	248	252	250	249	248	255
248	245	246	245	245	249	246	247	246	253


Histograma para Espesor
frecuencia
40
30
20
10
0
230
235
240
245
250
255
260
Espesor

Distintos problemas de inferencia

Dado un *modelo* *para los datos:*

$$X_1, X_2, \ldots, X_n$$

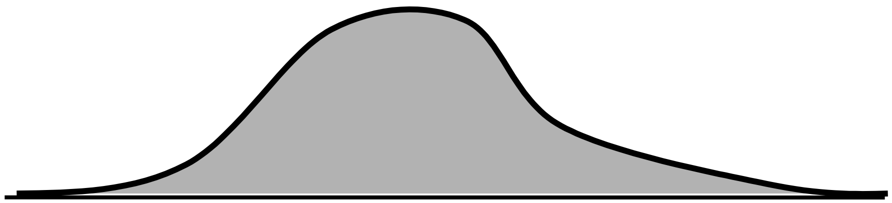


- *Estimar* μ y σ
- Dar un *intervalo de confianza* para μ y σ
- Elegir entre (*contraste de hipótesis*):

$$\mu \geq 250 \quad \text{o} \quad \mu < 250$$

- Comprobar la validez del modelo (*contraste de bondad de ajuste*).

Métodos de Estimación



$$X \rightarrow f_X(x, \theta_1, \theta_2, \ldots, \theta_r)$$

f_X conocida

Parámetros desconocidos

$$\theta_1, \theta_2, \ldots, \theta_r$$

Dada una muestra aleatoria simple de X

$$x_1, x_2, \ldots, x_n \quad \rightarrow \quad ¿\theta_1, \theta_2, \ldots, \theta_r?$$

1. Método de los momentos

2. Método de máxima verosimilitud

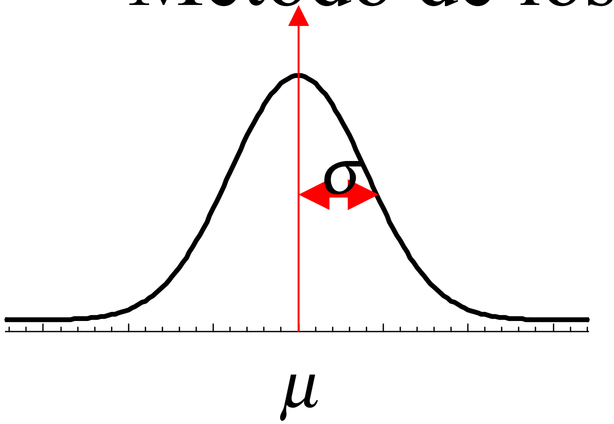
Métodos de los momentos

DATOS	VAR. ALEATORIA
$x_1, x_2, \ldots, x_n$	$f_X(x,\theta_1,\theta_2,...,\theta_r)$
$a_1 = \dfrac{\sum_{i=1}^n x_i}{n}$	$\alpha_1 = E[X] = g_1(\theta_1,\theta_2,...,\theta_r)$
$a_2 = \dfrac{\sum_{i=1}^n x_i^2}{n}$	$\alpha_2 = E[X^2] = g_2(\theta_1,\theta_2,...,\theta_r)$
$\vdots$	$\vdots$
$a_r = \dfrac{\sum_{i=1}^n x_i^r}{n}$	$\alpha_r = E[X^r] = g_r(\theta_1,\theta_2,...,\theta_r)$

Estimadores

$$\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r \rightarrow \begin{cases} g_1(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r) = a_1 \\ g_2(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r) = a_2 \\ \vdots \\ g_r(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r) = a_r \end{cases}$$

Método de los momentos: Distribución normal



$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\}, \quad x \in \Re$$

Parámetros : ¿ μ, σ ?

$$x_1, x_2, \ldots, x_n$$

$$a_1 = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$a_2 = \frac{\sum_{i=1}^{n} x_i^2}{n}$$

$$f_X(x, \mu, \sigma)$$

$$\alpha_1 = E[X] = \mu$$

$$\alpha_2 = E[X^2] = \sigma^2 + \mu^2$$

$$\hat{\mu} = \bar{x}$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 = s^2$$

Método de máxima verosimilitud (Introducción)

Una fuente radiactiva emite partículas según un proceso de Poisson con media λ desconocida. Durante 10 minutos se han contado el número de partículas emitidas:

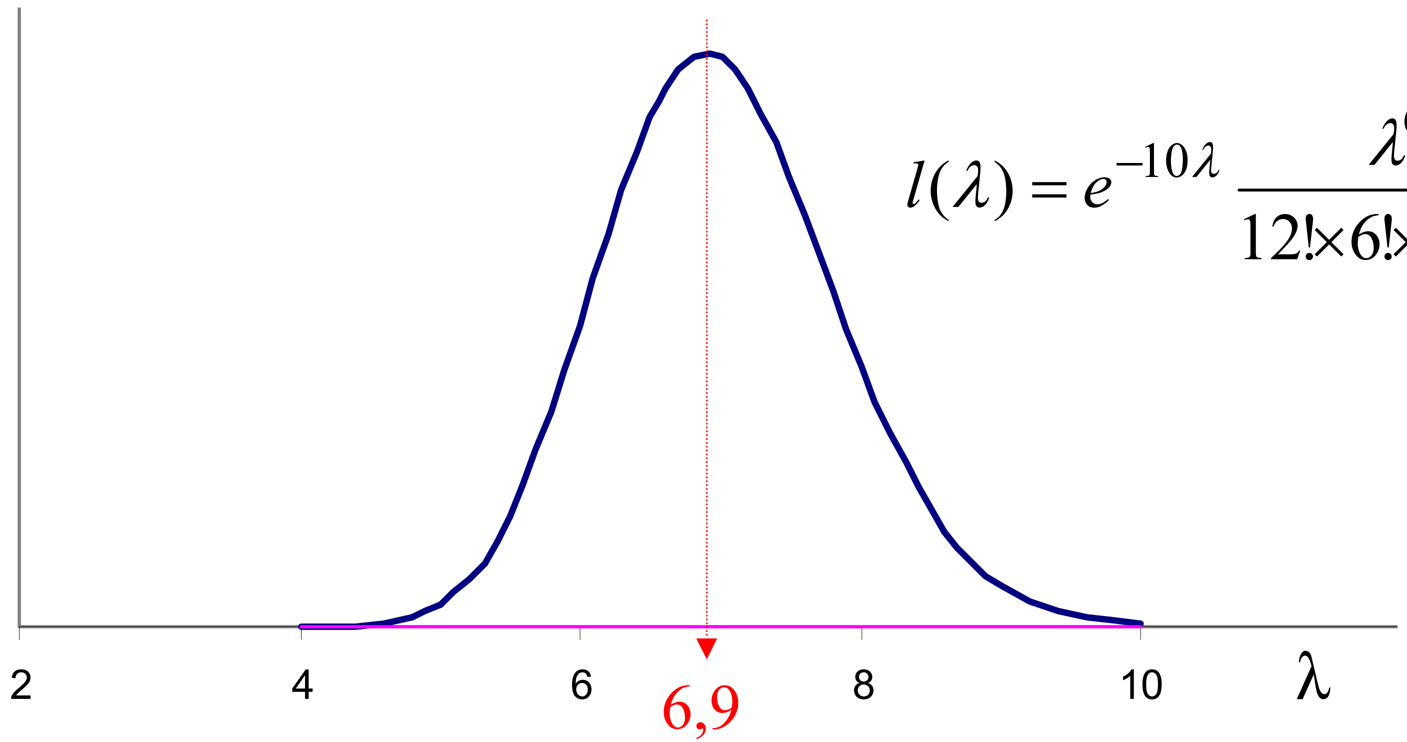
12, 6, 11, 3, 8, 5, 3, 9, 7, 5

$$P(X_1 = 12, X_2 = 6, \ldots, X_{10} = 5) = e^{-\lambda}\frac{\lambda^{12}}{12!} \times e^{-\lambda}\frac{\lambda^{6}}{6!} \times \cdots \times e^{-\lambda}\frac{\lambda^{5}}{5!}$$

$$= e^{-10\lambda}\frac{\lambda^{12+6+\cdots+5}}{12! \times 6! \times \cdots \times 5!} = e^{-10\lambda}\frac{\lambda^{69}}{12! \times 6! \times \cdots \times 5!}$$

$$\boxed{l(\lambda) = e^{-10\lambda}\frac{\lambda^{69}}{12! \times 6! \times \cdots \times 5!}}$$

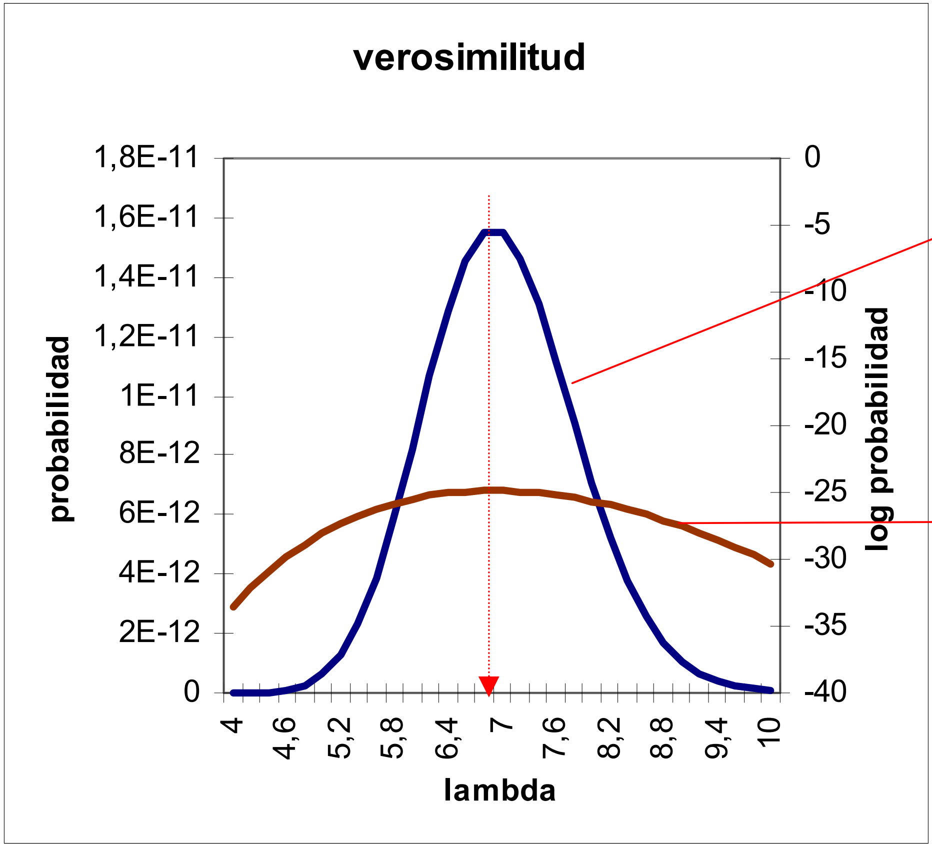
Función de verosimilitud



$$l(\lambda) = e^{-10\lambda} \frac{\lambda^{69}}{12! \times 6! \times \cdots \times 5!}$$

Estimador máximo-verosímil: $\hat{\lambda} = 6{,}9$

Función de verosimilitud



$$l(\lambda) = e^{-10\lambda} \frac{\lambda^{69}}{12! \times 6! \times \cdots \times 5!}$$

$$L(\lambda) = \log l(\lambda)$$

Estimador máximo-verosímil: $\hat{\lambda} = 6{,}9$

Estimación por máxima verosimilitud

$$X \to f_X(x, \theta_1, \theta_2, ..., \theta_r)$$

$$f_X \quad \text{conocida}$$

$$\text{Parámetros desconocidos} : \theta_1, \theta_2, ..., \theta_r$$

$$\text{Muestra aleatoria simple} : X_1, X_2, ..., X_n$$

$$\text{Distribución conjunta} :$$

$$f(x_1, x_2, ..., x_n; \theta_1, ..., \theta_r) =$$

$$= f_X(x_1, \theta_1, \theta_2, ..., \theta_r) f_X(x_2, \theta_1, \theta_2, ..., \theta_r) \cdots f_X(x_n, \theta_1, \theta_2, ..., \theta_r)$$

$$\log f(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \log f_X(x_i, \theta_1, \theta_2, ..., \theta_r) = L(\theta_1, \theta_2, ..., \theta_r)$$

$$\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_r \quad \Rightarrow \quad L(\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_r) = \max L(\theta_1, \theta_2, ..., \theta_r)$$

Máx. verosimilitud: Distribución normal

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad x \in \Re \quad \text{Parámetros}: ¿\,\mu, \sigma\,?$$

$X_1, X_2, ..., X_n$: muestra aleatoria simple

$$f(x_1, x_2, ..., x_n) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_1-\mu)^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_2-\mu)^2} \cdots \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_n-\mu)^2}$$

$$= \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2}$$

$$L(\mu, \sigma^2) = \log f(x_1, x_2, ..., x_n)$$

$$= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2$$

$$\max L: \quad \left\{\begin{array}{c} \dfrac{\partial L}{\partial \mu} = \dfrac{1}{\sigma^2}\sum_{i=1}^n (x_i - \hat{\mu}) = 0 \\ \\ \dfrac{\partial L}{\partial \sigma^2} = -\dfrac{n}{2}\dfrac{1}{\hat{\sigma}^2} + \dfrac{1}{2\hat{\sigma}^4}\sum_{i=1}^n (x_i-\hat{\mu})^2 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \hat{\mu} = \bar{x} \\ \hat{\sigma}^2 = s^2 \end{array}\right.$$

Máx. Verosimilitud: Poisson

$$P(X=x)=e^{-\lambda}\frac{\lambda^x}{x!}, x=0,1,2,\ldots \to \quad \text{Parámetro } \lambda$$

$$\text{Muestra aleatoria simple}: x_1, x_2, \ldots, x_n$$

$$p(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$$

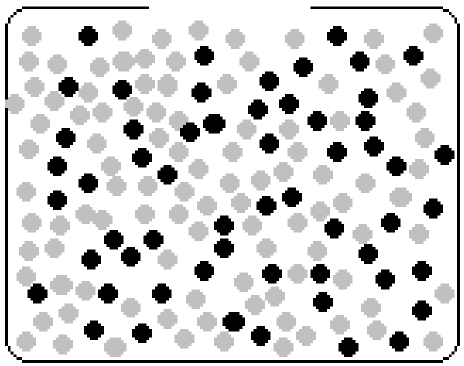
$$= e^{-\lambda}\frac{\lambda^{x_1}}{x_1!}e^{-\lambda}\frac{\lambda^{x_2}}{x_2!}\cdots e^{-\lambda}\frac{\lambda^{x_n}}{x_n!} = e^{-\lambda n}\frac{\lambda^{\Sigma x_i}}{x_1! x_2! \cdots x_n!}$$

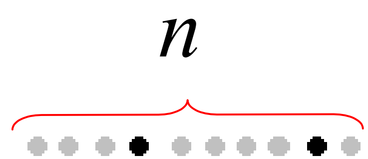
$$L(\lambda) = -\lambda n + (\Sigma x_i)\log\lambda - \sum \log x_i!$$

$$\max L(\lambda): \frac{dL(\lambda)}{d\lambda} = -n + \frac{\Sigma x_i}{\hat{\lambda}} = 0 \Rightarrow \hat{\lambda} = \bar{x}$$

Máx. verosimilitud: Binomial(n,p)

Proporción defectuosas = ¿ p ?





Muestra :

$x_1, x_2, \ldots, x_n$

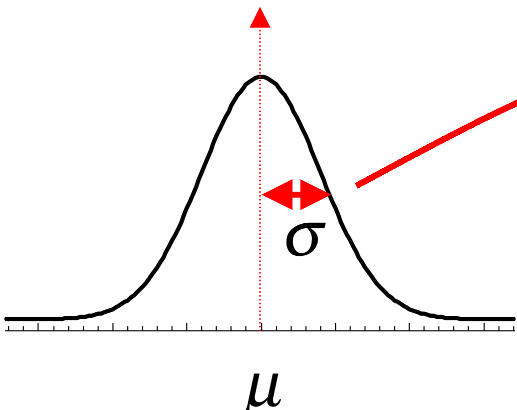
$$x_i = \begin{cases} 0 & \text{si es } \textbf{Aceptable} \\ 1 & \text{si es } \textbf{Defectuosa} \end{cases}$$

$$\begin{aligned} p(x_1, x_2, \ldots, x_n) &= P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) \\ &= P(X_1 = x_1) P(X_2 = x_2) \cdots P(X_n = x_n) \\ &= p^{\sum x_i} (1-p)^{n - \sum x_i} \\ &= p^r (1-p)^{n-r} \quad \text{donde } r = \sum x_i \text{ es el nº de defectuosas} \end{aligned}$$

$$L(p) = r \log p + (n-r) \log(1-p)$$

$$\frac{dL(p)}{dp} = \frac{r}{\hat{p}} - \frac{n-r}{1-\hat{p}} = 0 \quad \rightarrow \quad \boxed{\hat{p} = \frac{r}{n}}$$

Distribución de media (normal)



$$X \to N(\mu, \sigma)$$

$$X_1, X_2, \dots, X_n$$

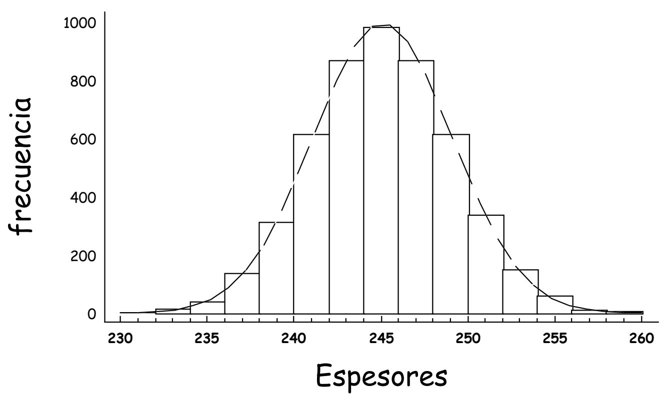
$$\boxed{\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}}$$

$$E[\overline{X}] = \frac{E[X_1] + E[X_2] + \cdots + E[X_n]}{n} = \frac{\mu + \mu + \cdots + \mu}{n} = \mu$$

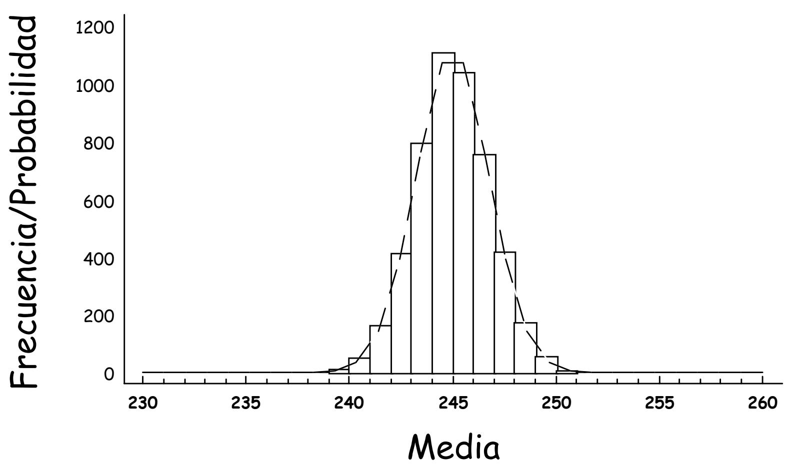
$$\mathrm{Var}[\overline{X}] = \frac{\mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \cdots + \mathrm{Var}[X_n]}{n^2} = \frac{\sigma^2 + \sigma^2 + \cdots + \sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$\boxed{\overline{X} \to N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)}$$

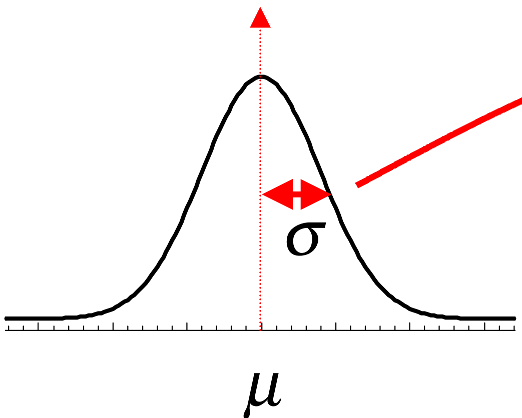
Histograma de Espesores



Distribución de la Media de 5 observaciones



Distribución de S² (Normal)



$$X \rightarrow N(\mu, \sigma)$$

$$X_1, X_2, \ldots, X_n$$

$$S^2 = \frac{(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \cdots + (X_n - \overline{X})^2}{n}$$

$$S^2 = \frac{1}{n}\sum_1^n (X_i - \overline{X})^2 = \frac{1}{n}\sum_1^n X_i^2 - \overline{X}^2$$

$$E[S^2] = \frac{1}{n}\Sigma\, E[X_i^2] - E[\overline{X}^2] = \frac{1}{n}\Sigma(\sigma^2 + \mu^2) - (\frac{\sigma^2}{n} + \mu^2) = \frac{n-1}{n}\sigma^2$$

$$\hat{S}^2 = \frac{(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \cdots + (X_n - \overline{X})^2}{n-1}$$

$$\hat{S}^2 = \frac{n}{n-1}S^2 \quad \Rightarrow \quad E[\hat{S}^2] = \sigma^2$$

Distribución χ^2

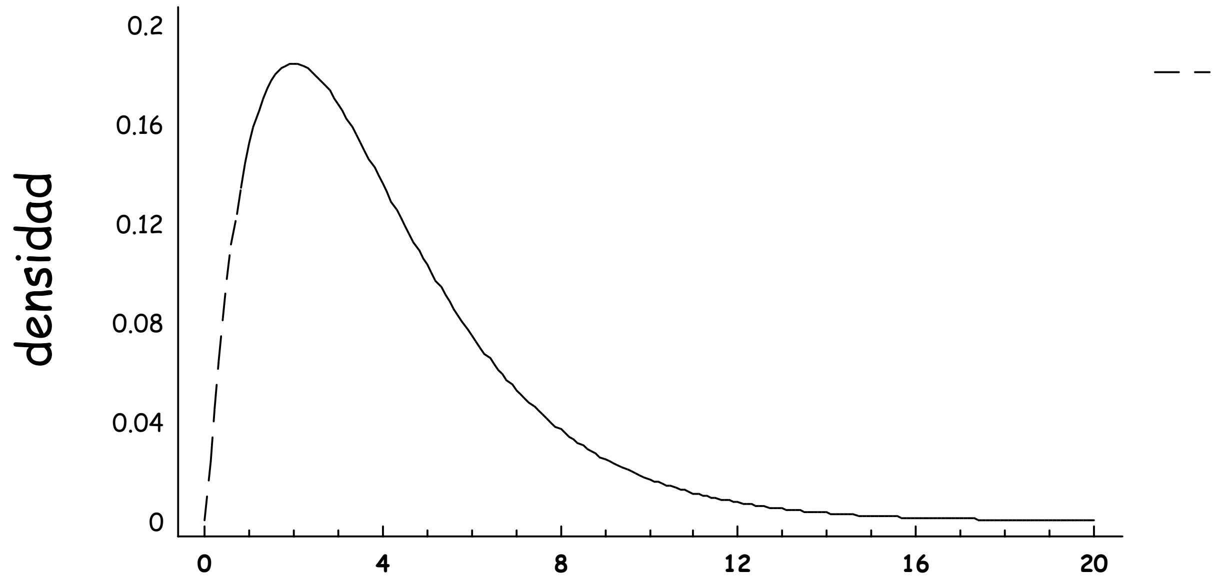
$$Z_1, Z_2, \ldots, Z_n \to N(0,1) \quad \text{independientes}$$

$$Z_1^2 + Z_2^2 + \cdots + Z_n^2 \to \chi_n^2$$

Propiedades

- ⊗ $E[\chi_n^2] = n$
- ⊗ $\mathrm{Var}[\chi_n^2] = 2n$
- ⊗ $\chi_n^2 + \chi_m^2 = \chi_{n+m}^2 \quad (\chi_n^2 \text{ y } \chi_m^2 \text{ indep.})$

Distribución Chi-cuadrado con 4 g.l.



Distribución de S^2 (Normal)

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \to \chi_n^2$$

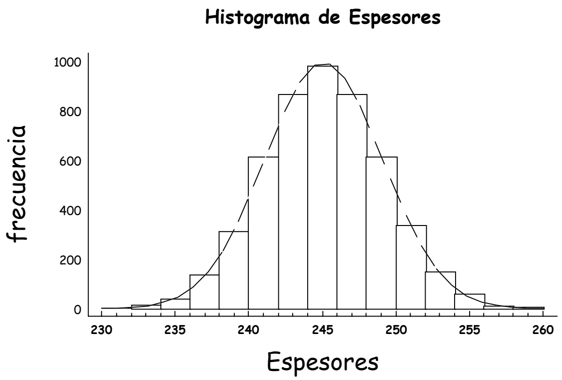
$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}(X_i-\overline{X}+\overline{X}-\mu)^2$$

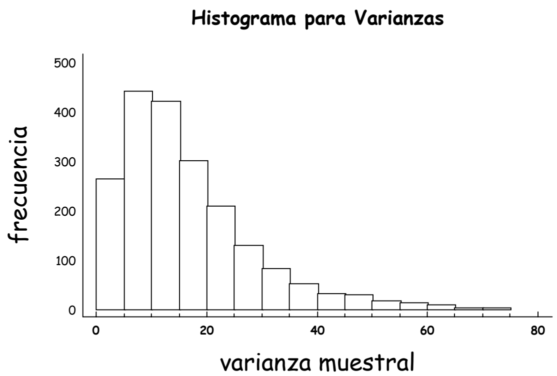
$$= \sum_{i=1}^{n}(X_i-\overline{X})^2 + \sum_{i=1}^{n}(\overline{X}-\mu)^2 + \overbrace{2\sum_{i=1}^{n}(X_i-\overline{X})(\overline{X}-\mu)}^{=\ 0}$$

$$= \sum_{i=1}^{n}(X_i-\overline{X})^2 + n(\overline{X}-\mu)^2$$

$$\underbrace{\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2}}_{\downarrow\ \chi_n^2} = \underbrace{\frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{\sigma^2}}_{\downarrow\ \chi_{n-1}^2} + \underbrace{\frac{(\overline{X}-\mu)^2}{\sigma^2/n}}_{\downarrow\ \chi_1^2}$$

$$\boxed{\begin{aligned} \frac{nS^2}{\sigma^2} &= \sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2 \xrightarrow{dist} \chi_{n-1}^2 \\ \frac{(n-1)\hat{S}^2}{\sigma^2} &= \sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2 \xrightarrow{dist} \chi_{n-1}^2 \end{aligned}}$$


Histograma de Espesores
frecuencia
1000
800
600
400
200
0
230
235
240
245
250
255
260
Espesores


Histograma para Varianzas
frecuencia
500
400
300
200
100
0
0
20
40
60
80
varianza muestral

Distribución de la media (general)

$$X_i \to f(x,\theta) : \mathrm{Var}[X_i] < \infty$$

$$\mathbf{X} = (X_1, X_2, ..., X_n)^T, \to \text{Vector de } n \text{ variables aleatorias independientes}$$

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$E[\overline{X}] = \frac{E[X_1] + E[X_2] + \cdots + E[X_n]}{n}$$

$$Var[\overline{X}] = \frac{Var[X_1] + Var[X_2] + \cdots + Var[X_n]}{n^2}$$

Si las variables tienen la misma media y varianza

$$\mu = E[X_i] \quad \forall i \qquad\qquad \sigma^2 = Var[X_i] \quad \forall i$$

$$n \to \infty$$

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n} \Rightarrow \begin{cases} E[\overline{X}] = \mu \\ Var[\overline{X}] = \dfrac{\sigma^2}{n} \end{cases} \Rightarrow \overline{X} \to N(\mu, \frac{\sigma}{\sqrt{n}})$$

Binomial

$$\text{Binomial}: X_1, X_2, ..., X_n$$

$$X_i = \begin{cases} 0 & \text{si es } \textbf{Aceptable} \\ 1 & \text{si es } \textbf{Defectuosa} \end{cases} \Rightarrow \begin{array}{c} E[X_i] = p \\ Var[X_i] = p(1-p) \end{array}$$

$$\hat{p} = \frac{X_1 + X_2 + \cdots + X_n}{n} \rightarrow \begin{cases} E[\hat{p}] = p \\ Var[\hat{p}] = \dfrac{p(1-p)}{n} \end{cases}$$

$$\boxed{\hat{p} \underset{n \to \infty}{\rightarrow} N(p, \sqrt{\frac{p(1-p)}{n}})}$$

Poisson

$$\text{Poisson} : X_1, X_2, \ldots, X_n$$

$$P(X_i = k) = e^{-\lambda} \frac{\lambda^k}{k!} \Rightarrow \begin{cases} E[X_i] = \lambda \\ Var[X_i] = \lambda \end{cases}$$

$$\hat{\lambda} = \frac{X_1 + X_2 + \cdots + X_n}{n} \rightarrow \begin{cases} E[\hat{\lambda}] = \lambda \\ Var[\hat{\lambda}] = \dfrac{\lambda}{n} \end{cases}$$

$$\boxed{\hat{\lambda} \underset{n \to \infty}{\longrightarrow} N(\lambda, \sqrt{\frac{\lambda}{n}})}$$

Propiedades de los estimadores

$$X_1, X_2, ..., X_n \text{ m.a.s de } f(x,\theta) : \hat{\theta} \equiv \hat{\theta}(X_1, X_2, ..., X_n)$$

- Centrados: $E[\hat{\theta}] = \theta$ $\quad (\text{Sesgo}[\hat{\theta}] = E[\hat{\theta}] - \theta)$
- Varianza mínima: $\forall \hat{\theta}' : \text{Var}[\hat{\theta}] \leq \text{Var}[\hat{\theta}']$
- *Error cuadrático medio* mínimo

$$\begin{aligned} \text{ECM}[\hat{\theta}] &= E[(\hat{\theta} - \theta)^2] \\ &= \text{Sesgo}^2(\hat{\theta}) + \text{Var}(\hat{\theta}) \end{aligned}$$

- Consistentes

$$\lim_{n \to \infty} E[\hat{\theta}] = \theta \quad \text{y} \quad \lim_{n \to \infty} \text{Var}[\hat{\theta}] = 0$$

Ejemplo 1

$$X_1, X_2, \ldots, X_n \text{ m.a.s de } N(\mu, \sigma) : \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\text{Es centrado} : E[\overline{X}] = \mu$$

Es de varianza mínima

Es consistente :

$$\lim_{n\to\infty} \mathrm{E}[\overline{X}] = \mu \quad \text{y} \quad \lim_{n\to\infty} \mathrm{Var}[\overline{X}] = 0$$

Ejemplo 2

$$X_1, X_2, \ldots, X_n \text{ m.a.s de } N(\mu,\sigma) : S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

$$\text{No es centrado}: E[S^2] = \frac{n-1}{n}\sigma^2$$

$$\text{Sesgo}(S^2) = -\frac{\sigma^2}{n}$$

Varianza :

$$\text{Var}\left[\frac{nS^2}{\acute{o}^2}\right] = 2(n-1) \Rightarrow Var[S^2] = \frac{2(n-1)}{n^2}\sigma^4$$

Es consistente :

$$\lim_{n\to\infty} \text{E}[S^2] = \sigma^2 \quad \text{y} \quad \lim_{n\to\infty} \text{Var}[S^2] = 0$$